

# Strange nonchaotic attractor in a dynamical system under periodic forcing

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We observe the occurrence of a strange nonchaotic attractor in a periodically driven two-dimensional map, formerly proposed as a neuron model and a sequence generator. We characterize this attractor through the study of the Lyapunov exponents, fractal dimension, autocorrelation function and power spectrum. The strange nonchaotic attractor in this model is a typical behavior, occupying a finite range of the parameter space.

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## I. INTRODUCTION

The strange nonchaotic attractor (SNA) is an object that has chaotic attractor features like fractal dimension and nondifferentiability (strangeness) but no exponential sensitivity to initial conditions, i.e., its largest Lyapunov exponent is nonpositive [1]. It has some analogies with trajectories that have been found in a study of the Frenkel-Kontorova model [2] and of the Chirikov-Taylor map [3]. Aubry has shown [4] that the incommensurate ground-states of the Frenkel-Kontorova model can undergo a *breaking of analyticity transition*, between a smooth and a fractal set. Shenker and Kadanoff [5] calculated the fractal power spectrum of the fractal trajectory that appears in the Chirikov-Taylor map after the breakup of a KAM-like surface a fractal power spectrum. This map can be related to an incommensurate driving of a nonlinear oscillator. Another example is the accumulation point of the period-doubling cascade of the logistic map [6]. However, this attractor occurs in a zero measure set in the parameter space. SNA as a typical behavior, a finite measure set in the parameter space of a model, has been found in the context of nonlinear quasiperiodic external forcing (i.e., the forcing of a signal with two incommensurate frequencies) [7]. The study of the SNA has also been recently connected with the localization problem [8,9].

A lot of work has been done in order to characterize the features of a SNA; its route of formation, autocorrelation function and power spectrum. Besides a period-doubling cascade, many different routes for the formation of an SNA have been proposed: (1) the collision between a period-doubled torus and its unstable parent torus [10]; (2) the progressive fractalization of a two-dimensional ergodic torus [11] and, (3) for systems with quasiperiodic tori in symmetric invariant subspaces, the loss of the transverse stability of a torus [12]. Another particular feature of the SNA is that its autocorrelation function does not decay with the time delay like that of the chaotic attractor. It can either be fractal or similar to the quasiperiodic case [13]. The power spectrum of a SNA can be singular continuous and in this case has

fractal features as discussed in many papers [13–18]

The aim of this paper is to answer the question: are SNAs restricted to quasiperiodically driven nonlinear systems? We believe that the answer is no. We studied a map (hereafter called YOS map) that has been proposed in the magnetic context to describe the behavior of an analog of the ANNNI model on the Bethe lattice [19,20] and which shows, for external periodic forcing, the occurrence of a SNA [21]. The same map exhibits typical features of a neuron (like activation threshold, nerve blocking and *rebound* behavior) and has been coined as a dynamical perceptron of second order (related to the dimension two of the map), because it corresponds to a two-layer recurrent neural network [22]. Independently, Kanter *et al.* [23] proposed a recurrence relation scheme known as a *sequence generator*, which is essentially the same map, generalized for any number of dimensions. This map can also be viewed as a discrete-time, nonlinear oscillator, for some values of the parameters.

Anishchenko *et al.* [18] have claimed that they found a SNA through periodic driving of a map, but Pikovsky *et al.* [24] have shown that it was a chaotic attractor with a tiny Lyapunov exponent. We show that Anishchenko *et al.*'s attractor is indeed strange chaotic, but ours is strange nonchaotic and related to quasiperiodic attractors.

This paper is organized as follows. Section II is dedicated to describing the map and its attractors, mainly the strange nonchaotic one. In section III we characterize this attractor through its Lyapunov exponents, fractal dimension, autocorrelation function and power spectrum. The conclusions are addressed to section IV.

## II. THE MAP AND ITS ATTRACTORS

The two-dimensional YOS map which this paper is concerned with is given by

$$\begin{aligned} x_{n+1} &= \tanh \left[ \frac{x_n - \kappa y_n + H(n)}{T} \right] \\ y_{n+1} &= x_n \end{aligned} \quad (1)$$

The YOS map was initially proposed [19] to model the mean magnetizations  $(x_n, y_n)$  of the  $(n^{th}; (n-1)^{th})$ -shells of a Bethe lattice, for an analog of the ANNNI model [25–27] in a constant magnetic field ( $H$  was independent of  $n$ ). Here, we extend it for a nonuniform field.  $T$  is the temperature and  $\kappa = -J_2/J_1$  where  $J_1(J_2)$  is the exchange coupling between nearest (next-nearest) neighbor spins on a Bethe lattice. Yokoi *et al.* obtained the phase diagram of this model at zero field [19]. Tragtenberg and Yokoi studied the effect of finite uniform field [20]. A sinusoidal wave  $H(n) = H_0 \cos(2\pi\omega n)$  is the particular form of  $H(n)$  we will adopt throughout this paper, representing a shell-dependent external field/input.

Kinouchi and Tragtenberg [22] studied the properties of the map as a neuron model, where  $x_n$  is the action potential of the neuron at time  $n$ .  $1/T$  and  $-\kappa/T$  are the weight factors for the two previous states of the neuron.  $H(n)/T$  is naturally defined as the external current as a function of the discrete time  $n$ . They showed that the map exhibits many neural features.

Kanter *et al.* [23] proposed the following real number sequence generator:

$$s_l = \tanh \left[ \beta \sum_{n=1}^N W_n s_{l-n} \right] \quad (2)$$

and studied this map in the context of time-series and neural networks.  $H(n)$  could be introduced for representing a time dependent input signal.

From the purely dynamical system point of view, the YOS map represents a nonlinear oscillator for the range of the parameters considered in this paper, and  $H(n)$  represents a time-dependent external input. The attractors of map YOS can be fixed points, Q-cycles (cycles of period  $Q$ ), quasiperiodic, chaotic or strange nonchaotic [19–21,28]. For the parameters  $\kappa = 1, T = 0.5, H_0 = 0$  the system oscillates with period 6 (see Fig. 1). But, for different values of  $H_0$  and  $\omega$ , keeping the same  $\kappa$  and  $T$ , the attractor becomes richer (see Fig. 2).

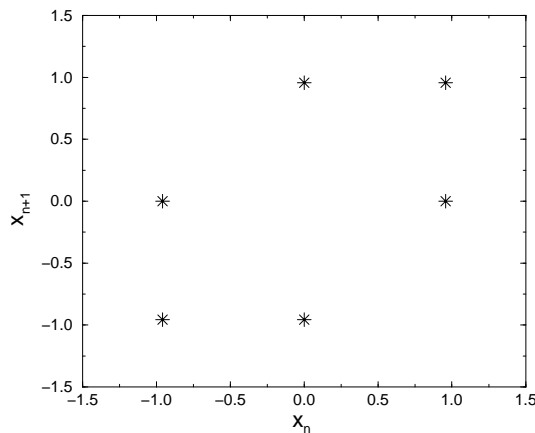


FIG. 1. First return map for the cycle of period 6 of the YOS map with  $\kappa = 1$ ,  $T = 0.5$  and  $H_0 = 0$ .

This kind of attractor can also be obtained for other values of  $\omega$ , and is therefore characteristic of this periodically driven nonlinear map. For others examples of SNA of this map see [28].

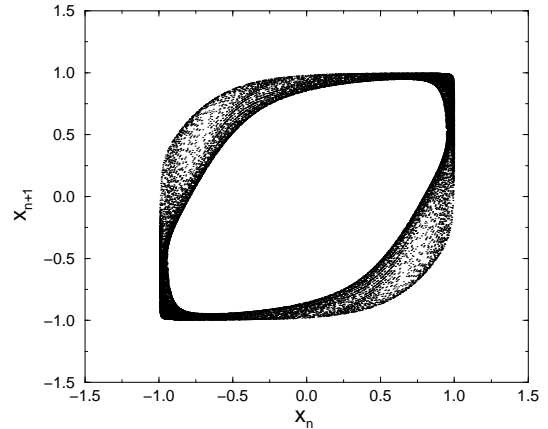


FIG. 2. First return map of the strange nonchaotic attractor of YOS map with  $\kappa = 1, T = 0.5, H_0 = 0.17$  and  $\omega = 0.14$ . The initial condition is  $(x_0, y_0) = (1, 1)$ . Here are represented 30 000 iterations, after neglecting the first 10,000.

### III. CHARACTERIZATION OF THE SNA

A SNA is a fractal object with no exponential sensitivity to initial conditions (the SNA at the accumulation point of the period-doubling bifurcations of the logistic map has null Lyapunov exponent but has polynomial sensitivity to initial conditions [29]). In order to characterize this attractor, we investigated the largest Lyapunov exponent, fractal dimension, autocorrelation function and power spectrum.

#### A. Lyapunov exponents: naïve and more accurate calculation

Before calculating the Lyapunov exponent of the attractor of Fig. 2, let us briefly discuss the sensitivity to initial conditions of one of the attractors studied by Anishchenko *et al.* [18].

They proposed a four-dimensional map made up by two asymmetrically coupled circle maps, with two coupling parameters ( $A$  and  $\gamma_2$ ). They argued that the for  $\gamma_2 = 0$  the system (1) can be a circle map with quasiperiodic forcing. A small value of  $\gamma_2$  will make it an autonomous four-dimensional map that could show a SNA.

This four-dimensional map is given by

$$\begin{aligned}
x_{n+1} &= x_n + \Omega_1 - \frac{K_1}{2\pi} \sin(2\pi x_n) + \gamma_1 y_n \\
&\quad + A \cos(2\pi u_n) \mod 1, \\
y_{n+1} &= \gamma_1 y_n - \frac{K_1}{2\pi} \sin(2\pi x_n), \\
u_{n+1} &= u_n + \Omega_2 - \frac{K_2}{2\pi} \sin(2\pi u_n) + \gamma_2 (y_n + v_n) \mod 1, \\
v_{n+1} &= \gamma_2 (y_n + v_n) - \frac{K_2}{2\pi} \sin(2\pi u_n).
\end{aligned} \tag{3}$$

For the set of parameters  $\Omega_1 = 0.5, \Omega_2 = (\sqrt{5} - 1)/2, K_2 = 0.03, A = 0.4, \gamma_1 = \gamma_2 = 0.01$  and  $K_1 = 0.8784$ , Anishchenko *et al.* claimed the attractor is strange nonchaotic. They found a null largest Lyapunov exponent within the numerical accuracy of the method they used.

A positive definite largest Lyapunov exponent corresponds to an exponential expansion of an hypercube of initial conditions in at least one of the directions of the phase space. In other terms, we can study the sign of the largest Lyapunov exponent of a map by studying the stretching and contraction of a hypercube of initial conditions. Here we present a simpler version of this procedure, studying only the evolution of the distance between trajectories generated by only two initial conditions.

We take two different sets of initial values of  $(x, y, u, v)$  and calculate the distance  $d(n)$  between the trajectories generated by each set, as the number of iterations is increased. That is perhaps the naïvest way to investigate the largest Lyapunov exponent of a map. The first set we take as  $x_0 = y_0 = u_0 = v_0 = 0.7$  and the second as  $x'_0 = y'_0 = u'_0 = v'_0 = (0.7 - 10^{-12})$ . Figure 3 represents the first 30 000 iterations of the evolution of  $d(n)$ .

We can see at first sight that the system is chaotic, since the distance between the trajectories with different initial conditions grows exponentially. A simple estimation of the slope of the rugged curve leads to  $(0.8 \pm 0.3)10^{-3}$  for the largest Lyapunov exponent  $\lambda_+$ , where we have assumed that the behavior of the distance is governed by this exponent and given by

$$d(n) \approx d(0) \exp(\lambda_+ n). \tag{4}$$

This result agrees with surprising accuracy with that obtained by Pikovsky and Feudel [24], using the Wolf-Swift-Swinney-Vastano algorithm [30].

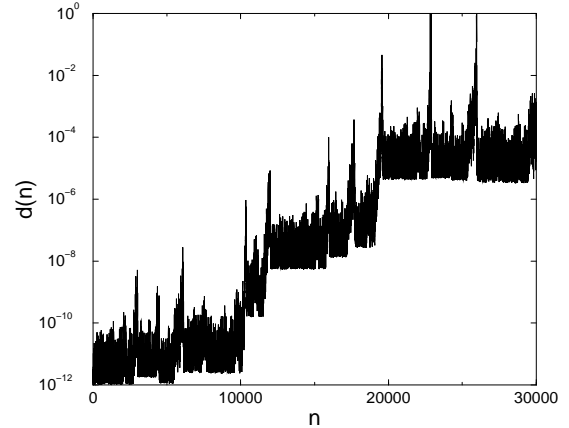


FIG. 3. Distance between attractors with different initial conditions  $d(n)$  as a function of the number of iterations  $n$ , for the map (3) for the parameters  $\Omega_1 = 0.5, \Omega_2 = \frac{\sqrt{5}-1}{2}, K_2 = 0.03, A = 0.4, \gamma_1 = \gamma_2 = 0.01$  and  $K_1 = 0.8784$ . The two set of initial conditions are  $x_0 = y_0 = u_0 = v_0 = 0.7$  and  $x'_0 = y'_0 = u'_0 = v'_0 = (0.7 - 10^{-12})$ . The largest Lyapunov exponent is  $\lambda_+ = (0.8 \pm 0.3)10^{-3}$ .

Then, this naïve method of checking the sensitivity of initial conditions seems to be powerful and we will use it as well as the more accurate method due to Eckmann-Kamphorst-Ruelle-Ciliberto (EKRC) [31] to calculate the largest Lyapunov exponent of the attractor of Fig. 2.

Fig. 4a exhibits some self-similarity in the behavior of  $d(n)$  as a function of  $n$ . We took many pairs of initial conditions such that the distances between the initial conditions from each pair were  $10^{-2}, 10^{-6}, 10^{-10}$  and  $10^{-14}$ . Then we calculated how  $d(n)$  vary for each pair as a function of the iterations, for the SNA of the Fig. 2. The result is represented in Fig. 4b. The various curves  $d(n) \times n$  have a scale invariant like behavior, i.e., for various values of the difference in initial conditions the evolution with the iterations is rather similar, preserving the form for different scales of distance. Moreover, none of the curves show exponential divergence, although they may vary within few orders of magnitude. It suggests a null largest Lyapunov exponent. We confirmed this using calculations based on the EKRC method shown below. This behavior (scale invariance in distance and zero largest Lyapunov exponent) is similar to that of a typical quasiperiodic attractor.

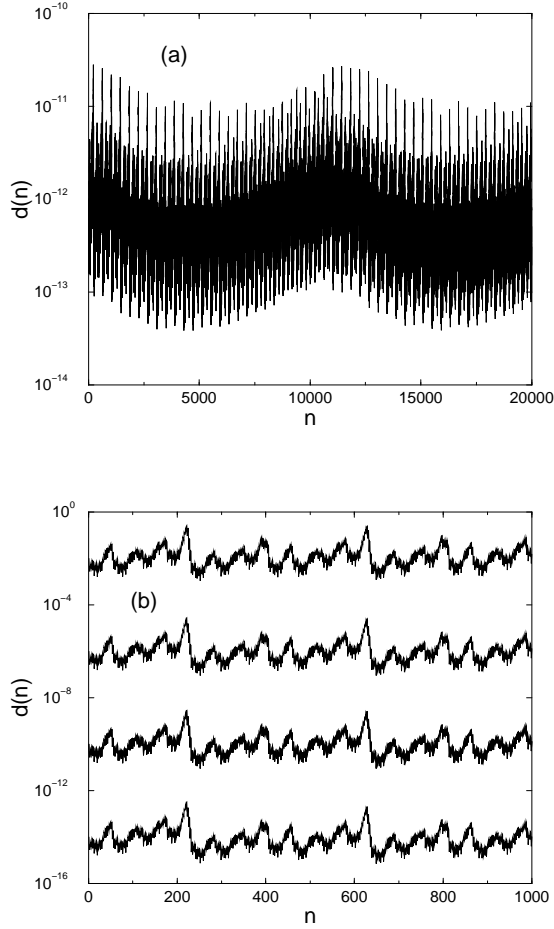


FIG. 4. Distance between trajectories with different initial conditions  $d(n)$  as a function of the number of iterations  $n$  for the SNA of the Fig. 2, when: (a) one initial condition is  $x_0 = y_0 = 1$  and the other is  $x'_0 = y'_0 = (1 - 10^{-12})$ . Notice the self-similar like structure of the peaks; (b) one initial condition is ever taken as  $x_0 = y_0 = 1$  and the other has the form  $x'_0 = y'_0 = (1 - \Delta)$ , for  $\Delta = 10^{-2}, 10^{-6}, 10^{-10}$  and  $10^{-14}$ . The distances between the trajectories remain limited even in the limit of large number of iterations, pointing out to a null largest Lyapunov exponent, and exhibiting a scale invariance.

Fig. 5 shows the behavior of the absolute value of the largest Lyapunov exponent approximants as a function of the number of iterations (neglecting the first 10,000), for the attractor of the Fig. 2. This attractor has the same shape for many initial conditions: we took  $(x_0, y_0) = (\pm 1, \pm 0.5, 0; \pm 1, \pm 0.5, 0)$ . The calculations were performed using the Eckmann-Kamphorst-Ruelle-Ciliberto method. They do indicate that the largest Lyapunov exponent is zero, since the absolute values of its approximants scale with the number of iterations  $n$  as  $|\lambda_+| \sim n^{-1}$ . Using the same method, we found  $\lambda_- = -0.709971 \pm 0.000001$ , for the smallest Lyapunov exponent.

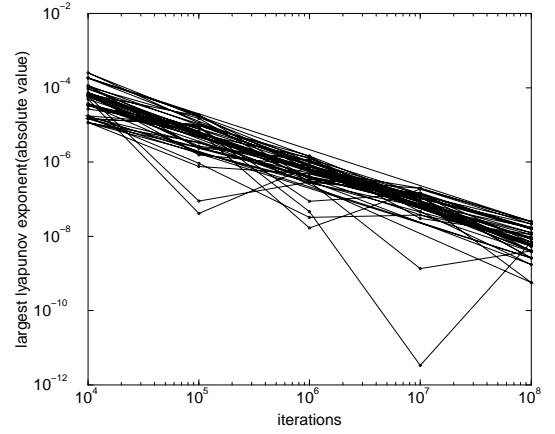


FIG. 5. Plot of the absolute values of the largest Lyapunov exponent approximants for the SNA of Fig. 2 as a function of the number of iterations  $n$  (the first  $10^4$  were neglected). The initial conditions considered were  $(x_0, y_0) = (\pm 1, \pm 0.5, 0; \pm 1, \pm 0.5, 0)$ . The absolute value of the approximants scales as  $|\lambda_+| \sim n^{-1}$ , indicating a zero largest Lyapunov exponent. The method used here is due to Eckmann-Kamphorst-Ruelle-Ciliberto [31].

## B. Fractal dimension

The SNA of the Fig. 2 is a complex geometrical object with a fractal Hausdorff dimension ( $D_F$ ). In order to find out this dimension we have used the box counting method [32]. The diagram with the number of boxes  $N(a)$  visited by the SNA of Fig. 2 as a function of the edge length  $a$  is shown in Fig. 6. The initial condition is  $(x_0, y_0) = (1, 1)$ , and the first  $10^4$  iterations were discarded. We considered the next  $10^9$  iterations, and box edges between  $a = 10^{-1}$  and  $a = 10^{-3}$ . Even with this number of iterations, we can see that box edges smaller than  $10^{-2.6}$  lead to artificially small values of the fractal dimension. However, a larger number of iterations were computationally prohibitive. The same behavior was observed in [33].

Fig. 7 has been constructed by taking ordered sets of four consecutive points of Fig. 6 and calculating the linear coefficient of the best straight line determined by them. Error bars follow from least squares fitting. The leftmost point of this figure represents the fitting of the four smallest values of  $\log(1/a)$ . The point of order 2 represents the fitting of the second to fifth point of Fig. 6, counting from the left to the right, and so on. Then, we conclude that the fractal dimension is  $D_F = 1.80 \pm 0.09$ . This is the same value found in Ref. [33] for the attractor of Grebogi *et al.* [7], but they considered this value quite uncertain (since it was obtained from just three points). In the same reference, Ding *et al.* used heuristic arguments to conjecture that  $D_F = 2$  and found no contradiction with the result they numerically found. But this value is definitely far from the value we obtained for

the attractor of the Fig. 2. The evidence points to the fractal character of this object.

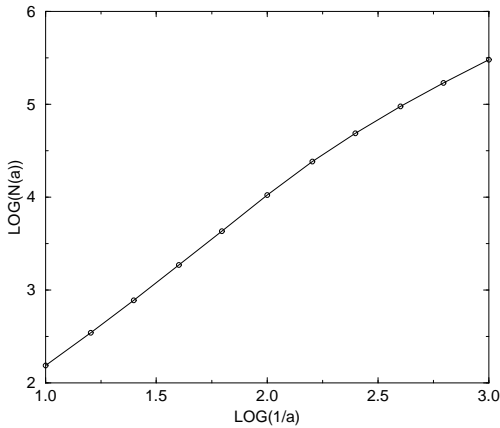


FIG. 6. Number of visited boxes  $N(a)$  as a function of the length of the box edge  $a$ , for the attractor of the Fig. 2. The first  $10^4$  iterations were neglected and the next  $10^9$  considered.

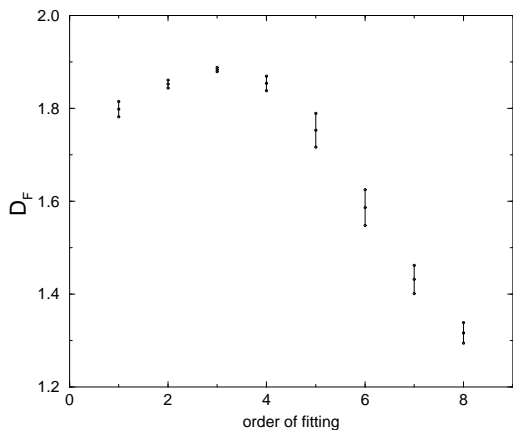


FIG. 7. Hausdorff dimension  $D_F$  of the attractor of Fig. 2 as a function of the order of fitting, for sets of four consecutive points of the Fig. 6. The order of fitting  $i$  denotes that the slope (and error bars) were calculated for the  $(i, i+1, i+2, i+3)^{th}$  points of Fig. 6 (the counting order begins from the left to the right).  $D_F$  is the value of the slope of the line determined by these point sets through least squares fitting.

### C. Autocorrelation function

The normalized autocorrelation function of an attractor  $\{x_n\}$  can be defined as

$$C(\tau) = \frac{\sum_{n=1}^N x(n)x(n+\tau)}{\sum_{n=1}^N x^2(n)}. \quad (5)$$

The calculation of the autocorrelation function can give clues about the nature of the attractor in question. Its fractal character can indicate the fractality of the attractor. However, when we are in the presence of a SNA we can observe at least two kinds of autocorrelation function: fractal or quasiperiodic [13].

Fig. 8 represents the autocorrelation function of the attractor of the Fig. 2, and is very similar to those related to quasiperiodic attractors, like that found in Ref. [13] for the strange nonchaotic attractor of the model C defined therein. We neglected the first  $10^4$  iterations and considered averages over the next  $10^5$  iterations.

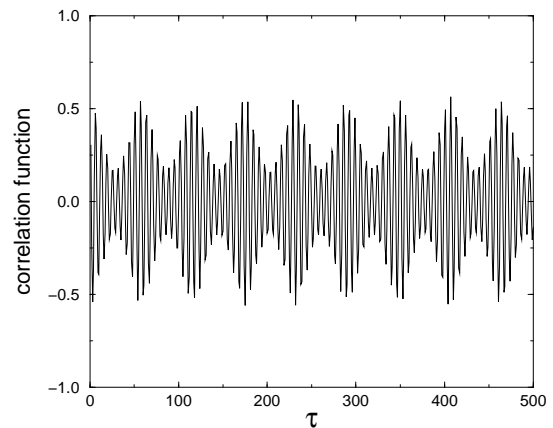


FIG. 8. Autocorrelation as a function of the time delay  $\tau$  for the attractor of the Fig. 2, which has the same form of the autocorrelation function of a quasiperiodic attractor. Here, we discarded the first  $10^4$  iterations and considered the next  $10^5$  iterations.

### D. Power spectrum

The first step in investigating the power spectrum of an attractor given by a sequence  $\{x_n\}$  is defining its discrete Fourier transform

$$s(w, N) = N^{-1/2} \sum_{n=1}^N x_n e^{i2\pi w n}. \quad (6)$$

Then, we can define the power spectrum of the attractor as:

$$P(w) = \lim_{N \rightarrow \infty} \langle |s(w, N)|^2 \rangle. \quad (7)$$

The power spectrum of periodic attractors consists of  $\delta$ -peaks at the harmonics of the fundamental frequency, whilst in the chaotic case the spectrum is continuous. For a quasiperiodic case characterized by two incommensurate frequencies  $\omega_1$  and  $\omega_2$ , the spectrum contains all the frequencies of the form  $n\omega_1 + m\omega_2$ .

Many works report power spectra of a SNA with singular continuous [5,13,15,17] character, like those found in some models of quasiperiodic lattices and quasiperiodically forced quantum systems [34–37]. This spectrum has a fractal appearance, where there are peaks weaker than  $\delta$ -functions distributed along a self-similar landscape.

Fig. 9 shows the power spectrum of the SNA of the Fig. 2. It has many scales of peaks, exhibiting a fractal appearance. We neglected the transient of the first  $10^4$  iterations and took the next  $10^4$ . The detailed study of the fractal character of this power spectrum as well as a renormalization group approach for it is the subject of a forthcoming publication.

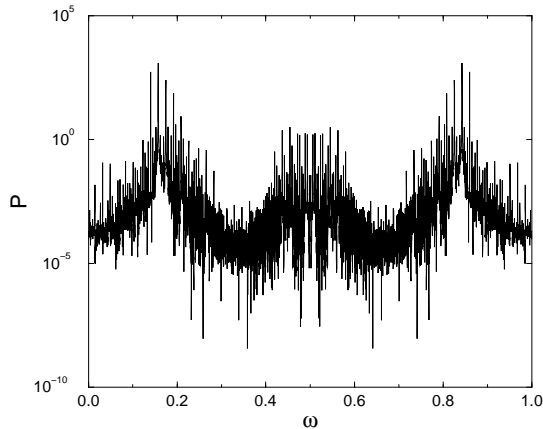


FIG. 9. Power spectrum of the attractor of Fig. 2, with a fractal appearance. The first  $10^4$  iterations were discarded and the following  $10^4$  considered. Notice the resonance at the driving frequency  $\omega = 0.14$ .

#### IV. CONCLUSIONS

We are able to show that a strange nonchaotic attractor can result from the dynamics of a periodically driven nonlinear oscillator, the YOS map. We have shown that a fractal object with zero largest Lyapunov exponent can emerge from this dynamics in a finite range of the parameters space. The Hausdorff dimension of this object is  $D_F = 1.80 \pm 0.09$ . Its correlation function oscillates like those of quasiperiodic attractors and its power spectrum has a fractal (or multifractal) appearance.

#### V. ACKNOWLEDGEMENTS

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